Entry of Magnetic Flux into a Magnetically Shielded Type-II Superconductor Filament

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(Dated: February 2, 2008)

In the framework of the London approximation the magnetic flux penetration into a type-II superconductor filament surrounded by a soft-magnet sheath and exposed to a transverse external magnetic field is studied. The lower transverse critical field as well as the critical field and the critical current of the first vortex nucleation at the superconductor/magnet interface are calculated on the basis of an exact solution for a vortex of arbitrary plane configuration. The Bean-Livingston barrier against the vortex nucleation is shown to strongly depend on the magnet sheath parameters.

PACS numbers: 74.25.Sv, 74.25.Op, 41.20.Gz, 84.71.Mn

I. INTRODUCTION

Hybrid systems composed of magnetic and superconducting materials attract much attention during last years in view of possibilities to improve superconductor critical parameters. There were conducted many experimental[1,2,3,4,5,6,7,8] and theoretical[9,10,11,12,13,14,15,16,17,18,19,20,21] studies of the heterostructures composed of superconductors (SC) and *ferromagnets* (FM). Diverse vortex configurations are generated in such structures due to the large intrinsic magnetic moments of the FM elements (magnetic dots[2,9,10,14,16,17] or inhomogeneities of the magnet layer[13,15,18]) and various transitions between them occur. The interaction of vortices with these intrinsic moments results in matching effects of a vortex lattice[1,3,7], spontaneous nucleation of vortices inside a superconductor layer[16,17,20,21], enhancement of vortex pinning[3,4,5,6,12] and increase of the critical magnetic field of a superconductor[7,11,17].

Much less attention has been attracted by heterostructures of superconductors and *soft magnets* (SM). Soft magnets, such as permalloy, pure iron, crioperm, etc., have, as a rule, sufficiently large values of the relative permeability μ, very narrow hysteresis loop and possess negligible remanent magnetization. Nevertheless, they may significantly improve superconductor performance by effective shielding from the external magnetic field as well as from the transport current self-field[22,23,24,25,26,27,28,29]. It was shown first theoretically[22,23] that the magnetic shielding may increase the critical current of a superconductor strip enhancing in this way its current-carrying capability both in the Meissner and in the partly flux-filled states. It was found also that such shielding can strongly reduce the transport AC losses in superconductor wires and tapes[24,25]. Strong current redistribution in superconductor strips due to bulk SM environments has been established recently by magneto-optics[26].

After the discovery of superconductivity in magnesium diboride[30] very intense investigations were carried out on superconducting MgB_2 wires sheathed in iron, which became ideal objects to explore the magnetic shielding effect due to simplicity of their fabrication[31,32]. As was observed in recent experiments, such structures exhibit enhanced superconducting critical currents over a wide range of the external magnetic field[31,33,34,35] as well as strong reduction of AC losses in the external field[36,37,38]. At the same time, a theoretical description of the influence of soft-magnet shielding on the current-carrying properties of type-II superconductor filaments is still lacking.

Recently, we have considered the flux-free Meissner state in a type-II superconductor filament surrounded by a circular soft-magnet sheath[39] and calculated the field distribution and the magnetic moment in this object. In the present paper we consider properties of the vortex state in the above sample when it is exposed to a transverse magnetic field and/or carries a transport current. In particular, we derive general expressions for the magnetic field of an arbitrary plane vortex and found the transverse lower critical field H_{c1} and the field of the first flux entry H_p. In addition to the experimental significance[31,32,33,34,35,36,37,38] of such SC/SM heterostructures, the system under consideration is simpler, from the theoretical point of view, than the strip geometry, because it allows one to exclude the strong influence of large geometrical factor on the superconducting response typical of the planar configurations[23,40,41,42]. In our consideration we will follow conceptually Ref.[43] where the lower critical field and the critical conditions for the first flux entry in a current-carrying type-II superconducting cylinder exposed to a transverse magnetic field have been established.

The paper is organized as follows. The theoretical model is presented in Sec. II. In Sec. III we derive the magnetic field distribution for a single vortex of an arbitrary plane shape and give a general expression for the self-energy of the vortex in a composite SC/SM filament. The dependences of vortex magnetic moment on the thickness and the relative permeability of the magnet sheath are discussed in Sec. IV. Further we found critical parameters of the SC/SM cylinder: the lower critical field H_{c1} is obtained in Sec. V and the conditions for the vortex loop nucleation on the SC/SM interface are established in Sec. VI. Our conclusions are presented in Sec. VII.

II. THEORETICAL MODEL

Let us consider an infinite type-II superconductor cylinder of radius R_1 enveloped in a coaxial magnetic sheath of thickness d with a relative magnetic permeability $\mu > 1$; the structure extended along the z axis of cylindrical coordinate system (ρ, φ, z) adapted to the cylinder (Fig. 1). A transverse magnetic field $\mathbf{H}_0$ is applied along the positive y direction and is asymptotically uniform at distances large compared to $R_2 = R_1 + d$. In our consideration we will neglect the remanent magnetization as well as both nonlinear behavior and conductivity of the magnetic layer so that the magnetic induction $\mathbf{B} = \mu_0\mu\mathbf{H}$ in the magnet and, therefore, a relative permeability μ is assumed the only characteristic of a homogeneous, isotropic SM sheath (μ_0 is the permeability of free space).

We start from the London equation for the magnetic induction $\mathbf{B}^{(1)}$ in the superconductor area[44]

$$\mathbf{B}^{(1)} + \lambda^2 \text{curl curl } \mathbf{B}^{(1)} = \mathbf{\Phi}, \quad \rho \leq R_1, \tag{1}$$

with the London penetration depth, λ, and the source function describing an arbitrary vortex

$$\mathbf{\Phi}(\mathbf{r}) = \Phi_0 \int d\mathbf{l}\, \delta(\mathbf{r} - \mathbf{l}), \tag{2}$$

where Φ_0 is the unit flux quantum, $\mathbf{r}$ is the position vector, $d\mathbf{l}$ is the flux-line element; the integration extends along the flux-line (vortex core). The magnetic field outside the superconductor denoted by $\mathbf{H}^{(2)}$ in the magnetic sheath and by $\mathbf{H}^{(3)}$ in the surrounding free space is described by Maxwell equations

$$\text{curl } \mathbf{H} = 0, \quad \rho \geq R_1, \tag{3}$$

$$\text{div } \mathbf{B} = 0, \tag{4}$$

the latter of which is valid in the whole space.

We imply the existence of an insulating, nonmagnetic layer of thickness much less than λ, d and R_1 between the superconductor and the magnet sheath (for example, such a layer was experimentally observed in MgB_2/Fe wires[35]). According to this assumption Eqs. (1-4) are provided with the following boundary conditions

$$B_n^{(1)} = \mu_0\mu H_n^{(2)}, \quad B_t^{(1)} = \mu_0 H_t^{(2)}; \tag{5a}$$

$$\mu H_n^{(2)} = H_n^{(3)}, \quad H_t^{(2)} = H_t^{(3)}, \tag{5b}$$

for the normal (n) and tangential (t) field components on the superconductor/magnet interface (5a) and on the outer magnet surface (5b), respectively. In addition, the field has to approach asymptotically the value of the external magnetic field $\mathbf{H}_0$.

The solution of Eqs. (1-4) may be represented as a superposition of the Meissner response $\mathbf{B}_M$, induced by $\mathbf{H}_0$ in absence of the magnetic vortex, and of the induction $\mathbf{b}$ of the vortex itself. The field $\mathbf{B}_M$ satisfies Eqs. (1-5) with $\mathbf{\Phi} = 0$ and has been found recently[39]. Therefore we may rewrite Eq. (1) in the form

$$\mathbf{b}^{(1)} + \lambda^2 \text{curl curl } \mathbf{b}^{(1)} = \mathbf{\Phi}, \quad \rho \leq R_1. \tag{6}$$

Taking into account that the field of the vortex is potential outside the superconductor and may be presented as $\mathbf{h} = -\boldsymbol{\nabla}\psi$ we obtain in this area, instead of Eqs. (3)-(4), the Laplace equation

$$\Delta\psi = 0, \quad \rho \geq R_1, \tag{7}$$

with $\psi \to 0$ at $\rho \to \infty$ and the boundary conditions (5) applying mutatis mutandis as well.

III. STRUCTURE OF A PLANE MAGNETIC VORTEX IN A SC/SM CYLINDRICAL SAMPLE

In the manner of Ref.[43], we look for the components of the vortex self-field in cylindrical coordinates $\mathbf{b}^{(1)} = \left(b_\rho^{(1)}, b_\varphi^{(1)}, b_z^{(1)}\right)$ and for the potential ψ using the Fourier transformation in the form

$$b^{j(1)}(\rho, \varphi, z) = \sum_m \exp(im\varphi) \int \frac{dk}{2\pi} b_{k,m}^{j(1)}(\rho) \exp(-ikz), \tag{8}$$

$$\psi(\rho,\varphi,z)=\sum_m \exp(im\varphi)\int\frac{dk}{2\pi}\psi_{k,m}(\rho)\exp(-ikz)\,, \tag{9}$$

where the index j assumes values ρ,φ,z. In terms of the Fourier amplitudes $b^{j(1)}_{k,m}$ and $\psi_{k,m}$ Eqs.(6)-(7) transform to the set of ordinary differential equations

$$\frac{\partial^2 b^{\rho(1)}_{k,m}}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial b^{\rho(1)}_{k,m}}{\partial\rho}-\left(Q^2+\frac{m^2+1}{\rho^2}\right)b^{\rho(1)}_{k,m}-\frac{2im}{\rho^2}b^{\varphi(1)}_{k,m}=-\frac{\Phi^{\rho}_{k,m}}{\lambda^2},\quad \rho\le R_1, \tag{10}$$

$$\frac{\partial^2 b^{\varphi(1)}_{k,m}}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial b^{\varphi(1)}_{k,m}}{\partial\rho}-\left(Q^2+\frac{m^2+1}{\rho^2}\right)b^{\varphi(1)}_{k,m}+\frac{2im}{\rho^2}b^{\rho(1)}_{k,m}=-\frac{\Phi^{\varphi}_{k,m}}{\lambda^2},\quad \rho\le R_1, \tag{11}$$

$$\frac{\partial^2 b^{z(1)}_{k,m}}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial b^{z(1)}_{k,m}}{\partial\rho}-\left(Q^2+\frac{m^2}{\rho^2}\right)b^{z(1)}_{k,m}=-\frac{\Phi^{z}_{k,m}}{\lambda^2},\quad \rho\le R_1, \tag{12}$$

$$\frac{\partial^2 \psi_{k,m}}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial \psi_{k,m}}{\partial\rho}-\left(k^2+\frac{m^2}{\rho^2}\right)\psi_{k,m}=0,\quad \rho\ge R_1, \tag{13}$$

with the boundary conditions

$$b^{\rho(1)}_{k,m}(R_1)=-\mu_0\mu\frac{\partial\psi^{(2)}_{k,m}(R_1)}{\partial\rho},\quad ib^{\varphi(1)}_{k,m}(R_1)=\mu_0\frac{m}{R}\psi^{(2)}_{k,m}(R_1)\,,\quad b^{z(1)}_{k,m}(R_1)=ik\mu_0\psi^{(2)}_{k,m}(R_1)\,, \tag{14}$$
$$\mu\frac{\partial\psi^{(2)}_{k,m}(R_2)}{\partial\rho}=\frac{\partial\psi^{(3)}_{k,m}(R_2)}{\partial\rho},\quad \psi^{(2)}_{k,m}(R_2)=\psi^{(3)}_{k,m}(R_2)\,,\quad \psi^{(3)}_{k,m}(\infty)=0.$$

Here the value $Q=\left(k^2+\lambda^{-2}\right)^{1/2}$ is introduced and the Fourier amplitudes $\Phi^{j}_{k,m}$ of the source function (2) are defined in the same manner as the field components in Eq. (8). Indices (2) and (3) in Eqs. (14) correspond to the areas of the magnetic sheath and the surrounding free space, respectively. Notice that Eqs. (10)-(12) are not identical to Eq. (6) since by their derivation we used the equality curl curl $\mathbf{b}^{(1)}=-\Delta\mathbf{b}^{(1)}$ which implies that div $\mathbf{b}^{(1)}=0$. Therefore the solutions of Eqs. (10)-(13) should also satisfy the latter constraint to be the solutions of Eqs.(6)-(7).

We consider below arbitrary configurations of a plane vortex lying in the plane $z=0$, so that $\Phi^{z}_{k,m}=0$. Upon the transformation $f^{\pm}_{k,m}=b^{\rho(1)}_{k,m}\pm ib^{\varphi(1)}_{k,m}$ the set of Eqs. (10)-(13) may be decoupled and solved it terms of the modified Bessel functions. We obtain the solutions regular at $\rho=0$:

$$\begin{aligned}\begin{pmatrix} b^{\rho(1)}_{k,m}\\ ib^{\varphi(1)}_{k,m}\end{pmatrix}=\frac{1}{2}\Bigg\{&I_{m+1}(Q\rho)\left[F^{+}_{k,m}-\int_{\rho}^{R_1}d\rho'\rho'\eta^{+}_{k,m}(\rho')K_{m+1}(Q\rho')\right]\\ &\pm I_{m-1}(Q\rho)\left[F^{-}_{k,m}-\int_{\rho}^{R_1}d\rho'\rho'\eta^{-}_{k,m}(\rho')K_{m-1}(Q\rho')\right]\\ &-K_{m+1}(Q\rho)\int_0^{\rho}d\rho'\rho'\eta^{+}_{k,m}(\rho')I_{m+1}(Q\rho')\mp K_{m-1}(Q\rho)\int_0^{\rho}d\rho'\rho'\eta^{-}_{k,m}(\rho')I_{m-1}(Q\rho')\Bigg\},\end{aligned} \tag{15}$$

$$b^{z(1)}_{k,m}=C_{k,m}I_m(Q\rho)\,, \tag{16}$$

$$\psi^{(2)}_{k,m}=\alpha_{k,m}I_m(|k|\rho)+\beta_{k,m}K_m(|k|\rho)\,, \tag{17}$$

$$\psi^{(3)}_{k,m} = \Psi_{k,m} K_m \left(|k|\,\rho\right), \tag{18}$$

where $\eta^{\pm}_{k,m}(\rho) = -\lambda^{-2}\left(\Phi^{\rho}_{k,m} \pm i\Phi^{\varphi}_{k,m}\right)$, I_ν and K_ν are the modified Bessel functions[45]. The coefficients in Eqs. (15)-(18) are found from Eqs. (14) and the constraint div $\mathbf{b}^{(1)} = 0$ and are given in Appendix A.

Before calculation of the physical properties of vortices of definite configurations and critical parameters of the system under consideration, we write here a general formula for the free energy of an arbitrary plane vortex in terms of the above presented solution. The self-energy of the system containing a single vortex takes a form (see Appendix B)

$$\begin{aligned} F &= \frac{1}{2\mu_0}\int_{\rho\le R_1} dV \left[\mathbf{b}^{(1)2} + \lambda^2\left(\text{curl }\mathbf{b}^{(1)}\right)^2\right] \\ &+ \frac{\mu_0\mu}{2}\int_{R_1\le\rho\le R_2} dV \ \left(\boldsymbol{\nabla}\psi^{(2)}\right)^2 + \frac{\mu_0}{2}\int_{\rho\ge R_2} dV \ \left(\boldsymbol{\nabla}\psi^{(3)}\right)^2 \\ &= \frac{1}{2\mu_0}\int_{\rho\le R_1} dV \ \mathbf{b}^{(1)}\boldsymbol{\Phi} + \frac{R_1}{2}\sum_m \int dk \ \psi^{(2)}_{k,m}(R_1)\,\Phi^{\rho}_{-k,-m}(R_1)\,. \end{aligned} \tag{19}$$

It is interesting to note that this expression is identical to the corresponding formula for the self-energy of a vortex in a non-shielded superconducting cylinder (see Eq. (21) in Ref.[43]); the presence of a magnet is accounted for implicitly by the actual form of the potential $\psi^{(2)}$ and of the magnetic induction $\mathbf{b}^{(1)}$.

The above described solution (15-18) exhibits a proper transformation to the case of an isolated superconductor[43] by setting $\mu = 1$. Together with the expression for the vortex self-energy (19) it may be applied to any single or multiple plane vortex configurations. Among others it allows one to investigate, by special choice of the vortex shape, the lower critical field and the magnetic flux entry in shielded superconductors carrying a transport current and/or subjected to an external magnetic field, the problems considered below.

IV. MAGNETIC MOMENT OF AN ARBITRARY PLANE VORTEX LOOP

In this section we consider a magnetic moment of a magnetically shielded wire, an important measurable characteristic which is also necessary for evaluation of critical parameters of the SC/SM heterostructure. The magnetic moment projection on the field $\mathbf{H}_0$ direction (see Fig. 1) consists of two parts, presenting contributions from the superconductor and from the magnetic sheath as follows

$$M_y = M^{(1)}_y + M^{(2)}_y = \frac{1}{2}\int_{\rho\le R_1} dV\,[\boldsymbol{\rho}\times\mathbf{j}]_y - (\mu-1)\int_{R_1\le\rho\le R_2} dV \ \left(\boldsymbol{\nabla}\psi^{(2)}\right)_y, \tag{20}$$

which may be reduced to forms

$$M^{(1)}_y = -\frac{2\pi}{\mu_0}R_1^2 b^{\varphi(1)}_{0,1}(R_1) + \frac{2\pi i}{\mu_0}\int_0^{R_1} d\rho\ \rho\left(b^{\rho(1)}_{0,1} - i b^{\varphi(1)}_{0,1}\right), \tag{21}$$

$$M^{(2)}_y = 2\pi i\,(\mu-1)\left[R_2\psi^{(2)}_{0,1}(R_2) - R_1\psi^{(2)}_{0,1}(R_1)\right]. \tag{22}$$

Let us consider now an arbitrary vortex loop lying in the plane $z = 0$ and penetrating the superconductor to the depth of r from the surface. For simplicity we also suppose that the loop is symmetric with respect to x axis and, therefore, we will describe its specific form by some smooth monotonic function $\varphi = \chi(\rho)$. Let us denote $R_1 - r$ as the least value of radius ρ for which the loop exists, then $\chi(R_1 - r) = 0$. With function $\chi(\rho)$ defined in this way the Fourier amplitudes of the source function (2) read[43]:

$$\begin{aligned} \Phi^{\rho}_{k,m} &= \frac{\Phi_0}{i\pi\rho}\sin\left[m\chi(\rho)\right]\theta\left(\rho - R_1 + r\right), \quad \Phi^{z}_{k,m} = 0, \\ \Phi^{\varphi}_{k,m} &= \frac{\Phi_0}{\pi}\frac{d\chi(\rho)}{d\rho}\cos\left[m\chi(\rho)\right]\theta\left(\rho - R_1 + r\right). \end{aligned} \tag{23}$$

Then, upon the substitution of the amplitudes $\Phi_{0,1}^{\rho}$ and $\Phi_{0,1}^{\varphi}$ in the general expressions (15), (17) and (20-22), we obtain the magnetic moment of the loop:

$$M_y = M_y^0 \frac{2\mu + \left(\mu^2+1\right)(d/R_1)(2+d/R_1)}{2\mu + (\mu+1)\left[1+(\mu-1)\, I_1'\left(R_1/\lambda\right)/I_0\left(R_1/\lambda\right)\right](d/R_1)(2+d/R_1)}, \tag{24}$$

where the prime denotes the derivative of the Bessel function with respect to its argument, and

$$M_y^0 = \frac{4\Phi_0}{\mu_0\lambda}\frac{1}{I_0\left(R_1/\lambda\right)}\int\limits_{R_1-r}^{R_1} d\rho\rho I_1\left(\frac{\rho}{\lambda}\right)\sin\left[\chi\left(\rho\right)\right] \tag{25}$$

is the magnetic moment for the unshielded superconductor filament[43] (notice that the formula (A4) in Ref.[43] may be reduced to the above form). As it follows from Eq. (24), the dependence of the magnetic moment on the relative permeability μ and the thickness d of the magnet sheath is universal in the sense that it does not depend on the specific form of the vortex. It is interesting to note that the magnetic moment may be factorized as in Eq. (24) though the contributions of the superconductor and of the magnet sheath are superimposed in the definition (20).

The $M_y\left(\mu,d\right)$ dependence is shown in Fig. 2 for two different values of superconductor radius: $R_1 = 10\lambda$ and $R_1 = \lambda$. One can see that this dependence is different for thick and thin superconductors. For enough thick superconductor (see Fig. 2,a) the moment M_y reveals a minimum as a function of μ for any fixed d. In the limit of $R_1 \gg \lambda$ the $M_y\left(\mu,d\right)$ dependence is described by the expression

$$M_y = M_y^0 \frac{2\mu + \left(\mu^2+1\right)(d/R_1)(2+d/R_1)}{2\mu + \mu(\mu+1)(d/R_1)(2+d/R_1)}. \tag{26}$$

The minimum value of the moment is reached at the permeability

$$\mu_* = 1 + \left[2 + \frac{2}{(d/R_1)(2+d/R_1)}\right]^{1/2} \tag{27}$$

and equals

$$M_y^* = M_y^0 \frac{\mu_*^2 - 1}{\mu_*^2}. \tag{28}$$

In the case of very thick magnet sheath ($d \gg R_1$) the lowest value of the moment is $M_y^* \simeq 0.83 M_y^0$.

In the limit of thin superconductor $R_1 \ll \lambda$ the magnetic moment M_y monotonically increases with increase of μ or d (see Fig. 2,b) and this dependence is described by the expression

$$M_y = M_y^0 \frac{2\left[2\mu + \left(\mu^2+1\right)(d/R_1)(2+d/R_1)\right]}{4\mu + (\mu+1)^2(d/R_1)(2+d/R_1)}. \tag{29}$$

Let us note that in both limiting cases the factor accounting for the magnet sheath does not depend on λ.

For investigation of critical parameters of the SC/SM heterostructure we must specify a form of the vortex which will be done in the next sections.

V. LOWER CRITICAL FIELD H_{c1} OF A SC/SM CYLINDER IN A TRANSVERSE EXTERNAL FIELD

To obtain the value of the lower critical magnetic field we consider now the case of a vortex taking a stable position in the center of the sample, namely directed along the cylinder diameter parallel to the applied field $\mathbf{H}_0$ (Fig. 3). This position is analogous to that of the straight vortex located deep inside a bulk superconductor cylinder parallel to the external field which energy defines the lower critical field for bulk samples[44]. The central location of the vortex apparently leads to a local minimum of the Gibbs free energy of the system

$$G = F - \mathbf{M}\mathbf{H}_0, \tag{30}$$

where F is the self-energy (19) and $\mathbf{M}$ is the magnetic moment of the sample due to the presence of the vortex (24). Vanishing of the energy (30) defines the value of the lower critical field H_{c1} at which the vortex becomes firstly

energetically favorable deep inside the superconductor (notice that, in this case, the Meissner contribution to the energy is constant and may be omitted).

For the vortex lying along the cylinder diameter the Fourier amplitudes of the source function (2) read[43]:

$$\Phi^{\rho}_{k,m} = \frac{\Phi_0}{i\pi\rho}\sin\frac{\pi m}{2}, \quad \rho \le R_1, \qquad (31)$$
$$\Phi^{\varphi}_{k,m} = \Phi^{z}_{k,m} = 0.$$

Upon the substitution of Eq. (31) into the solution (15-18), one can find approximate expressions for the self-energy of the vortex (19) in two limiting cases:

$$F \simeq \frac{\Phi_0^2}{2\pi\mu_0\lambda^2}R_1\left[\ln\frac{4R_1}{e\xi} + \frac{6}{\pi} + O\left(R_1/\lambda\right)\right], \qquad \xi \ll R_1 \ll \lambda, \qquad (32)$$

$$F \simeq \frac{\Phi_0^2}{2\pi\mu_0\lambda^2}R_1\left[\ln\frac{\lambda}{\xi} - \gamma + O\left(\lambda/R_1\right)\right], \qquad R_1 \gg \lambda, \qquad (33)$$

where the divergence of the energy at large momentum k, usual in the London theory[44], is cut at the scale $k \sim 1/\xi$ with the superconductor coherence length ξ. One can see that in both cases of thin ($R_1 \ll \lambda$) and thick ($R_1 \gg \lambda$) superconductors the self-energy does not depend on the characteristics of magnet sheath even in terms of the order of small parameters R_1/λ and, respectively, λ/R_1. The dependence on μ may appear only in terms of the higher orders of that small parameters. Therefore, the magnet sheath virtually does not influence the self-energy of the vortex.

The magnetic moment of the sample is defined by Eq. (24), where M_y^0 is easily obtained from Eq. (25) with $\chi\left(\rho\right) \equiv \pi/2$ (in this case the integration over ρ starts from 0):

$$M_y^0 = \frac{2\pi\Phi_0 R_1}{\mu_0}\left[\frac{L_0\left(R_1/\lambda\right)}{I_0\left(R_1/\lambda\right)}I_1\left(R_1/\lambda\right) - L_1\left(R_1/\lambda\right)\right], \qquad (34)$$

where L_ν is the modified Struve function. Notice that, contrary to the self-energy, the magnetic moment M_y strongly depends on μ and d. Finally, from Eq. (30) we easily obtain the following expression for the lower critical field:

$$H_{c1} = H_{c1}^0\frac{2\mu + (\mu+1)\left[1 + (\mu-1)\, I_1'\left(R_1/\lambda\right)/I_0\left(R_1/\lambda\right)\right]\left(d/R_1\right)\left(2 + d/R_1\right)}{2\mu + \left(\mu^2+1\right)\left(d/R_1\right)\left(2 + d/R_1\right)}, \qquad (35)$$

where H_{c1}^0 is the transverse lower critical field of the non-sheathed sample, calculated in Ref.[43]. Notice that by virtue of definition the $H_{c1}\left(\mu, d\right)$ dependence is inverted to that of the magnetic moment (24).

The dependences of H_{c1} on μ and d are shown in Fig. 4 for the same values of radius of the superconductor as in Fig. 2. One can see that for enough large radii this dependence is nonmonotonic and reveals the region of magnet permeability values where $H_{c1} > H_{c1}^0$. For $R_1 = 10\lambda$ (Fig. 4,a) the lower critical field is enhanced up to 10% and this enhancement grows with increase of R_1. In the limiting case $R_1 \gg \lambda$ the $H_{c1}\left(\mu, d\right)$ dependence is described by the expression

$$H_{c1} = H_{c1}^0\frac{\mu\left[2 + (\mu+1)\left(d/R_1\right)\left(2 + d/R_1\right)\right]}{2\mu + \left(\mu^2+1\right)\left(d/R_1\right)\left(2 + d/R_1\right)}. \qquad (36)$$

The maximum value of H_{c1} is reached at any fixed value of the magnet layer thickness d for the permeability μ_* (27) and equals

$$H_{c1} = H_{c1}^0\frac{\mu_*^2}{\mu_*^2 - 1}, \qquad (37)$$

taking on the largest value $H_{c1} \cong 1.2H_{c1}^0$ at $d \gg R_1$. With decrease of R_1 this effect disappears and for $R_1 \lesssim \lambda$ the presence of magnet sheath depresses the lower critical field (see Fig. 4,b). In the limit of thin superconductor core $R_1 \ll \lambda$ the critical field is described by the asymptotic expression:

$$H_{c1} = \frac{H_{c1}^0}{2}\frac{4\mu + (\mu+1)^2\left(d/R_1\right)\left(2 + d/R_1\right)}{2\mu + \left(\mu^2+1\right)\left(d/R_1\right)\left(2 + d/R_1\right)}. \qquad (38)$$

A practical conclusion here is that a cylindrical magnet sheath has a detrimental effect on superconductivity in thin superconductor wires of radius less than λ facilitating vortex phase at lower magnetic fields. On the other hand, the magnetic coating of thick superconductors with radius much larger than λ allows optimization of the sheath parameters d and μ in reasonable ranges leading to the moderate enhancement of the lower critical field.

VI. VORTEX LOOP NUCLEATION AT THE SC/SM INTERFACE (THE BEAN-LIVINGSTON BARRIER)

In the non-shielded type-II superconductor sample exposed to a transverse magnetic field the entry of magnetic flux starts with the small loop nucleation at the sample surface[46,47,48,49] when the surface Bean-Livingston barrier[50] is overcome. It is evident that in the SC/SM system concerned the similar process of vortex loop nucleation takes place at the interface between the superconducting core and the magnet sheath (Fig. 5). However, due to the magnetization of magnetic medium, the nucleation of vortex loop at the SC/SM interface may differ from this on the uncovered SC cylinder surface studied earlier[43].

A. Nucleation of vortex loop in a transverse magnetic field

The Bean-Livingston barrier is a result of competition between the attraction of the vortex to the boundary and the repulsive Lorentz force exerted upon the vortex by the Meissner current. To evaluate the critical field of the first vortex loop penetration into the SC cylinder it is convenient to present the Gibbs energy of the system as a sum of the vortex loop free energy and the work of the external source of the magnetic field calculated as the work of the Meissner current[51],

$$G = F - \Delta W_H . \tag{39}$$

We consider a small semicircle loop of radius $a \ll \lambda$ defined in Ref.[43] by the source function (23) with

$$\chi(\rho) = \phi_t \theta(R_1 - \rho)\, \theta\left(\rho - \sqrt{R_1^2 - a^2}\right) + \widetilde{\chi}(\rho)\, \theta(\rho - R_1 + a)\, \theta\left(\sqrt{R_1^2 - a^2} - \rho\right), \tag{40}$$

where $\sin\phi_t = a/R_1$ and $\cos\widetilde{\chi}(\rho) = \left(R_1^2 + \rho^2 - a^2\right)/2R_1\rho$. Substituting the amplitudes $\Phi^j_{k,m}$ into the general expression for energy (19) with magnetic field components from Eqs. (15)-(18) we find that the self-energy of the vortex loop for two limiting cases of thick ($R_1 \gg \lambda$) and thin ($R_1 \ll \lambda$) superconducting core coincides, in the main approximation, with the result for the unshielded SC cylinder[43],

$$F \simeq \frac{\Phi_0^2}{4\pi\mu_0\lambda^2}\pi a \ln\frac{a}{\xi}, \qquad \xi \ll a \ll \lambda, \tag{41}$$

and does not depend on parameters of magnetic sheath as well as in the case of the straight vortex along the diameter (see Sec. V). The dependence of the loop self-energy on permeability μ appears only in higher orders of the small parameter a/λ. Because of complexity of the general expressions (15)-(18) it is hard to exactly derive these terms. Fortunately, it is sufficient here to estimate the difference of the loop energy in the case under consideration from the case of unshielded superconductor. This difference reaches its maximum in the limit of an infinitely large μ (see, for example, Ref.[40]) and is less than the main approximation (41) by the factor of the order a/λ. Therefore, we conclude that the self-energy of the vortex loop is not affected substantially by presence of the magnet.

The above paradoxical result, that the influence of the magnet sheath on the vortex self-energy is inessential even for large permeabilities, can be explained in the following way. Apart from the major contribution to the vortex energy (41) proportional to its length, the full energy (19) includes a contribution of the magnetized sheath. The boundary condition (5a) requires the continuity of the normal component of magnetic induction $B_n^{(1)} = \mu_0\mu H_n^{(2)}$. At the same time, the total flux of magnetic induction is fixed by the flux quantization in a superconductor. Accordingly, a typical magnetic induction value in the magnet is $B \sim \Phi_0/\lambda^2$, whereas the magnetic field in the magnet $H \sim \Phi_0/\mu_0\mu\lambda^2$ is suppressed by the large μ. Therefore, the interaction energy proportional to $B \cdot H$ is μ times reduced comparing with the vacuum case. Notice that this conclusion is valid for the vortex loop nucleated at a SC/SM interface of arbitrary form.

Now let us calculate the second part of the Gibbs energy (39). In the geometry of Fig. 1 the Meissner current is perpendicular to the loop plane and almost constant in the small loop region of size $a \ll \lambda$. In this case the work of the Meissner current when the loop expands from the radius $r = 0$ to a reads simply as

$$\Delta W_H = \Phi_0 \int_0^a dr \int dl\; j_M = \frac{1}{2}\Phi_0 \pi j_s a^2, \tag{42}$$

where j_s is a value of the screening current in a place of the loop entry. The Meissner current in superconducting cylinder has the only z component[39]

$$j_z(\rho,\varphi) = \frac{4\mu}{D}\frac{H_0}{\lambda} I_1(\rho/\lambda)\cos\varphi, \tag{43}$$

$$D = \mu I_0\left(\frac{R_1}{\lambda}\right)\left[\mu + 1 - \frac{\mu - 1}{(1 + d/R_1)^2}\right] - \frac{I_1(R_1/\lambda)}{R_1/\lambda}\left(\mu^2 - 1\right)\frac{(d/R_1)(2 + d/R_1)}{(1 + d/R_1)^2}, \tag{44}$$

and the maximum magnitude of screening current

$$j_s = \frac{4\mu}{D}\frac{H_0}{\lambda} I_1(R_1/\lambda) \tag{45}$$

is achieved at the equatorial lines $\varphi = 0$ and $\varphi = \pi$ where a vortex nucleates most probably.

The Gibbs energy of the vortex loop nucleating at $\varphi = 0$ renormalized by presence of the magnet,

$$G = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2}\pi a \ln\frac{a}{\xi} - \frac{1}{2}\Phi_0\pi j_s a^2, \tag{46}$$

grows with the radius a from zero until it achieves a maximum at some critical radius value a_m defined by the relation $\partial G/\partial a = 0$. If the fluctuation vortex reaches this size, further loop expansion becomes irreversible and the vortex entry proceeds. Depending on a sample surface quality vortex penetration may occur at different values of the critical radius from the region $\xi < a_m < \lambda$ where the formula (46) applies. The lower value corresponds to the case of the ideal surface when the nucleation occurs at the scale of the vortex core, ξ. The opposite limit describes a rough surface with the typical imperfection size δ of the order of λ or larger. In general, a field of the first flux penetration, H_p, defined by the condition $a_m = \min(\delta,\lambda)$, is given by

$$H_p = \frac{H_p^0}{4}\left\{\frac{I_0(R_1/\lambda)}{I_1(R_1/\lambda)}\left[\mu + 1 - \frac{\mu - 1}{(1 + d/R_1)^2}\right] - \frac{\lambda}{\mu R_1}\left(\mu^2 - 1\right)\frac{(d/R_1)(2 + d/R_1)}{(1 + d/R_1)^2}\right\}, \tag{47}$$

where $H_p^0 = (\Phi_0/4\pi\mu_0\lambda a_m)\ln(e a_m/\xi)$ is a field of the first flux penetration at a flat superconductor/vacuum boundary adopting values between the lower and the thermodynamic critical fields of the bulk materials[44].

In Fig. 6 we present the dependence of the field H_p on the relative permeability μ and the thickness d for the radii of superconductor $R_1 = 10\lambda$ and $R_1 = \lambda$. One can see that this dependence is monotonic for both cases of thick and thin superconductor, contrary to the same dependences of the transverse lower critical field H_{c1}, and differs only by the scale of magnitudes. With increase of μ at fixed thickness d the $H_p(\mu)$ dependence approximates to the linear one. A monotonic behavior is also demonstrated by the $H_p(d)$ dependence with increase of d at fixed permeability μ. Considering practically interesting case $R_1 \gg \lambda$ we write down

$$H_p = \frac{H_p^0}{4}\left[\mu + 1 - \frac{\mu - 1}{(1 + d/R_1)^2}\right]. \tag{48}$$

In the opposite limit $R_1 \ll \lambda$ we obtain

$$H_p = H_p^0\frac{\lambda}{4\mu R_1}\left[(\mu + 1)^2 - \frac{(\mu - 1)^2}{(1 + d/R_1)^2}\right]. \tag{49}$$

B. Nucleation of vortex loop in presence of transport current

Next, we consider the situation when the superconductor carries also a transport current. The appearance of a total transport current J results in an additional angle independent z-component of the current density[52,53]

$$j_{tr}(\rho) = \frac{J}{2\pi R_1\lambda}\frac{I_0(\rho/\lambda)}{I_1(R_1/\lambda)}, \tag{50}$$

which is superimposed on the screening current (43). Similarly to the latter, the transport current density remains constant within the loop and equal to the surface value

$$j_{s,tr} = \frac{J}{2\pi R_1 \lambda} \frac{I_0\left(R_1/\lambda\right)}{I_1\left(R_1/\lambda\right)}. \tag{51}$$

This surface magnitude should be simply added to the maximum value of the screening current (45) and substituted in the Gibbs energy (46). Using the criterion $a_m = \min\left(\delta, \lambda\right)$ to define the critical current of the first flux penetration, J_c, we find the average density of the transport critical current, $j_c = J_c/\pi R^2$, for $H_0 < H_p$

$$j_c(H_0) = \frac{2I_1\left(R_1/\lambda\right)}{R_1 I_0\left(R_1/\lambda\right)} \left[H_p^0 - \frac{4\mu H_0}{D} I_1\left(R_1/\lambda\right)\right]. \tag{52}$$

In the limits of thin and thick superconductor core we obtain, respectively,

$$j_c(H_0) = \frac{H_p^0}{\lambda} - \frac{4\mu H_0 R_1}{\lambda^2} \left[\left(\mu+1\right)^2 - \frac{\left(\mu-1\right)^2}{\left(1+d/R_1\right)^2}\right]^{-1}, \qquad R_1 \ll \lambda, \tag{53}$$

$$j_c(H_0) = \frac{2H_p^0}{R_1} - \frac{8H_0}{R_1} \left[\mu+1 - \frac{\mu-1}{\left(1+d/R_1\right)^2}\right]^{-1}, \qquad R_1 \gg \lambda. \tag{54}$$

One can see that in view of relatively large values of the permeability in soft-magnet materials the field dependence of j_c remains linear up to the field $H_0 \lesssim H_p$, that corresponds to the experimental situation (see, for example, Ref.[31]).

For estimation at practically interesting temperature of 32 K we take a reasonable $\mu \cong 50$ for Fe and $d/R_1 \cong 1/2$ from Refs.[31,33,34] and thermodynamic parameters of MgB_2 from Ref.[54] which gives possible values of $j_c(0)$ between $7.0 \cdot 10^3$ and $4.4 \cdot 10^4$ A/cm^2 and of $\mu_0 H_p$ between 0.16 and 1.02 T. The field dependence of $j_c(H_0)$ remains very weak up to the fields comparable with H_p. In view of the relatively low values of critical temperature, Ginzburg-Landau parameter and anisotropy of the polycrystalline MgB_2 (Ref.[54]) we assume that thermally activated penetration through the surface barrier[47,48] is negligible. These estimates are in a good agreement with the results of Refs.[31,33,34].

VII. CONCLUSIONS

To summarize, we have described the structure of an arbitrary plane vortex in a type-II superconductor cylindrical filament covered by a coaxial soft magnet sheath when it is exposed to an external transverse magnetic field and carries a transport current. We have derived general expressions for the magnetic field components, self-energy and magnetic moment of the vortex. Using these expressions, we have established that the self-energy of the vortex lying along the sample diameter as well as that of the small vortex loop nucleated at the interface between the superconductor and the magnet is practically independent on the parameters of magnet sheath, i.e. its permeability and thickness. This paradoxical property is due to the phenomenon of flux quantization in the superconductor.

We have found that the dependence of the magnetic moment of an arbitrary plane vortex on both permeability and thickness of the magnet sheath is not sensitive to the specific form of the vortex. At the same time this dependence is qualitatively different for thick and thin superconductors. For the samples with radius much larger than λ it reveals the minimum at some value of the permeability for any fixed thickness of the sheath whereas for thin superconductors having a radius less than λ the magnetic moment of vortex monotonously increases when the permeability and/or sheath thickness increase. Such a behavior of the magnetic moment causes the inverted dependence of the transverse lower critical field, the exact expression of which was derived. This field has a maximum at some value of the relative permeability for any fixed thickness of the magnet sheath for the case of the thick superconductor, while it decreases monotonously with both permeability and magnet thickness increase for the case of the thin superconductor.

We have considered also how the magnet sheath changes the conditions for the flux penetration in the superconductor (the Bean-Livingston barrier) when the sample is exposed to the external transverse magnetic field and/or carries the transport current. The expressions for the critical field and the critical current of the first flux penetration have been derived. We found that, due to presence of the magnet sheath, the critical field of the first flux entry can be strongly enhanced. In contrast to the lower critical field, it strongly increases in both thick and thin superconductor cases, when the magnetic permeability of the sheath is large. Obtained results have shown that, due to the magnet sheath, the Meissner state in the superconductor filament can be effectively preserved in wide region of the external magnetic field (or transport current) magnitudes.

Notice also, that the above studied modification of the Bean-Livingston barrier is not reduced to the shielding effect of the magnetic sheath solely[25,36,37,38] because the latter does not account for the magnetic flux expelling from the filament itself. For example, in the limit $R_1 \gg \lambda, d$ the maximum field on the superconductor surface amounts to $H_{max} = 2H_0/(1+\mu d/R_1)$ while for the shielding of the normal core it is $H_{max} = H_0/(1+\mu d/2R_1)$ (see Ref.[55]). The difference between these two values may be substantial if the parameter $\mu d/R_1$ is not large. The observed range of external magnetic fields, where the critical transport current remains virtually field-independent, exceeds substantially the characteristic field of effective shielding by the magnet sheath alone which was discussed in Ref.[33]. On the other hand, estimations of the field for the first flux entry, H_p, correlates well with the region of weak field dependence of the critical current in MgB_2/Fe cylindrical wires[31,33]. Reduced flux penetration in the superconductor in this field region entails naturally reduced AC losses which was also observed in Refs.[36,37,38].

Finally, we have considered the simple model where the relative permeability is a sole material characteristic of the magnet and neglected its possible dependence on the applied field which may be important[39]. Nevertheless, obtained results are in good qualitative and quantitative agreement with the existent experiments and could be used to optimize the superconducting and current-carrying parameters of SC/SM heterostructures.

Acknowledgments

We are grateful to H. Rauh, H. C. Freyhardt, Ch. Jooss, A. V. Pan, J. Horvat, S. X. Dou, M. D. Sumption, F. M. Peeters, A. Gurevich and V. Vinokur for stimulating discussions. This work was supported by a research grant of the German Research Foundation (DFG). The support from the ESF VORTEX Program is also acknowledged.

APPENDIX A: THE COEFFICIENTS IN EQUATIONS (15)-(18)

The coefficients in Eqs. (15)-(18) read

$$\alpha_{k,m} = A_1 \, \frac{B_+ + B_- - B_3}{\mu_0 \Delta}, \quad \beta_{k,m} = A_2 \, \frac{B_+ + B_- - B_3}{\mu_0 \Delta}, \tag{A1}$$

$$\begin{aligned} F^{\pm}_{k,m} = B_{\pm} - \alpha_{k,m}\mu_0 \left|k\right| \left[\frac{I_{m\pm1}\left(\left|k\right| R_1\right)}{I_{m\pm1}\left(QR_1\right)} + (\mu-1)\,\frac{I'_m\left(\left|k\right| R_1\right)}{I_{m\pm1}\left(QR_1\right)} \right] \\ - \beta_{k,m}\mu_0 \left|k\right| \left[-\frac{K_{m\pm1}\left(\left|k\right| R_1\right)}{I_{m\pm1}\left(QR_1\right)} + (\mu-1)\,\frac{K'_m\left(\left|k\right| R_1\right)}{I_{m\pm1}\left(QR_1\right)} \right], \end{aligned} \tag{A2}$$

$$C_{k,m} = ik\mu_0 \left[\alpha_{k,m} \, \frac{I_m\left(\left|k\right| R_1\right)}{I_m\left(QR_1\right)} + \beta_{k,m} \, \frac{K_m\left(\left|k\right| R_1\right)}{I_m\left(QR_1\right)} \right], \tag{A3}$$

$$\Psi_{k,m} = \frac{1}{\left|k\right| R_2 I_m\left(\left|k\right| R_2\right) I'_m\left(\left|k\right| R_2\right)} \, \frac{B_+ + B_- - B_3}{\mu_0 \Delta}, \tag{A4}$$

where

$$A_1 = \frac{(1-\mu)}{\mu} \, \frac{K_m\left(\left|k\right| R_2\right)}{I_m\left(\left|k\right| R_2\right)} \, \frac{K'_m\left(\left|k\right| R_2\right)}{I'_m\left(\left|k\right| R_2\right)}, \quad A_2 = \frac{K_m\left(\left|k\right| R_2\right)}{I_m\left(\left|k\right| R_2\right)} - \frac{1}{\mu}\frac{K'_m\left(\left|k\right| R_2\right)}{I'_m\left(\left|k\right| R_2\right)}, \tag{A5}$$

$$\begin{aligned} \Delta = A_1 \Bigg\{ \left|k\right| \left[\frac{I_{m+1}\left(\left|k\right| R_1\right)}{I_{m+1}\left(QR_1\right)} + \frac{I_{m-1}\left(\left|k\right| R_1\right)}{I_{m-1}\left(QR_1\right)} \right] - \frac{2k^2}{Q}\frac{I_m\left(\left|k\right| R_1\right)}{I_m\left(QR_1\right)} \\ + (\mu-1)\left|k\right| I'_m\left(\left|k\right| R_1\right) \left[\frac{1}{I_{m+1}\left(QR_1\right)} + \frac{1}{I_{m-1}\left(QR_1\right)} \right] \Bigg\} \\ + A_2 \Bigg\{ -\left|k\right| \left[\frac{K_{m+1}\left(\left|k\right| R_1\right)}{I_{m+1}\left(QR_1\right)} + \frac{K_{m-1}\left(\left|k\right| R_1\right)}{I_{m-1}\left(QR_1\right)} \right] - \frac{2k^2}{Q}\frac{K_m\left(\left|k\right| R_1\right)}{I_m\left(QR_1\right)} \\ + (\mu-1)\left|k\right| K'_m\left(\left|k\right| R_1\right) \left[\frac{1}{I_{m+1}\left(QR_1\right)} + \frac{1}{I_{m-1}\left(QR_1\right)} \right] \Bigg\}, \end{aligned} \tag{A6}$$

$$B_{\pm}=\frac{K_{m\pm1}\left(QR_1\right)}{I_{m\pm1}\left(QR_1\right)}\int\limits_0^{R_1}d\rho\ \rho\eta^{\pm}_{k,m}\left(\rho\right)I_{m\pm1}\left(Q\rho\right),\quad B_3=\frac{2R_1}{\lambda^2 Q}K_m\left(QR_1\right)\Phi^{\rho}_{k,m}\left(R_1\right),\tag{A7}$$

the prime denotes the derivative of the Bessel function with respect to its argument.

APPENDIX B: THE FREE ENERGY OF A HYBRID SC/SM STRUCTURE

Let us calculate the excess free energy due to presence of vortices in an arbitrary hybrid system composed of the superconductor and insulating soft-magnet components with respect to the energy of that system in the flux-free Meissner state. Taking into consideration the potential nature of the magnetic field outside the superconductor we write this energy, using the London approximation, in the following way

$$F=\frac{1}{2\mu_0}\int dV^{(1)}\left[\mathbf{b}^{(1)2}+\lambda^2\left(\text{curl }\mathbf{b}^{(1)}\right)^2\right]+\frac{1}{2}\int dV^{(2)}\ \mathbf{B}^{(2)}\mathbf{h}^{(2)}+\frac{\mu_0}{2}\int dV^{(3)}\left(\mathbf{h}^{(3)}\right)^2,\tag{B1}$$

where $\mathbf{b}^{(1)}$ is the magnetic field of vortices in the superconductor, $\mathbf{h}^{(2)}$ and $\mathbf{h}^{(3)}$ are the magnetic fields in the magnet media and in the surrounding free space, respectively, $\mathbf{B}^{(2)}=\mu_0\mu\mathbf{h}^{(2)}$ is the magnetic induction in the magnet media. With the vector identity $\text{div}(\mathbf{a}\times\mathbf{c})=\mathbf{c}\text{ curl }\mathbf{a}-\mathbf{a}\text{ curl }\mathbf{c}$ and with the London equation (6) in the superconducting region the first term of Eq. (B1) becomes

$$\begin{aligned}F^{(1)}&=\frac{1}{2\mu_0}\int dV^{(1)}\left\{\mathbf{b}^{(1)2}+\lambda^2\left[\text{curl curl }\mathbf{b}^{(1)}+\text{div}\left(\mathbf{b}^{(1)}\times\text{curl }\mathbf{b}^{(1)}\right)\right]\right\}\\&=\frac{1}{2\mu_0}\int dV^{(1)}\ \mathbf{b}^{(1)}\mathbf{\Phi}+\frac{\lambda^2}{2\mu_0}\int dS^{(1)}\mathbf{n}^{(1)}\left(\mathbf{b}^{(1)}\times\text{curl }\mathbf{b}^{(1)}\right)\Big|_{S^{(1)}},\end{aligned}\tag{B2}$$

The second and third terms of Eq. (B1) are transformed, with the definition $\mathbf{h}^{(2,3)}=-\boldsymbol{\nabla}\psi^{(2,3)}$ and with the identity $\text{div}(\psi\mathbf{a})=\psi\text{div }\mathbf{a}+\mathbf{a}\boldsymbol{\nabla}\psi$, in the following surface integrals

$$\begin{aligned}F^{(2)}&=-\frac{1}{2}\int dV^{(2)}\left[\text{div}\left(\psi^{(2)}\mathbf{B}^{(2)}\right)-\psi^{(2)}\text{div }\mathbf{B}^{(2)}\right]\\&=-\frac{1}{2}\int dS^{(2)}\mathbf{n}^{(2)}\left(\psi^{(2)}\mathbf{B}^{(2)}\right)\Big|_{S^{(2)}},\end{aligned}\tag{B3}$$

$$F^{(3)}=-\frac{\mu_0}{2}\int dV^{(3)}\mathbf{h}^{(3)}\boldsymbol{\nabla}\psi^{(3)}=-\frac{\mu_0}{2}\int dS^{(3)}\mathbf{n}^{(3)}\left(\psi^{(3)}\mathbf{h}^{(3)}\right)\Big|_{S^{(3)}}.\tag{B4}$$

In Eqs. (B2-B4) $\mathbf{n}^{(j)}$ denotes the outer normal to the surface $S^{(j)}$ of the corresponding region. Next we assume that the field $\mathbf{h}^{(3)}$ decreases sufficiently fast, so that the part of surface integral (B4) corresponding to the integration at $\mathbf{r}\to\infty$ vanishes. The rest of this surface integral, i.e. integral over the magnet outer surface, is compensated by the corresponding part of the surface integral (B3) by virtue of the boundary condition (5b) and we have

$$\begin{aligned}F&=\frac{1}{2\mu_0}\int dV^{(1)}\ \mathbf{b}^{(1)}\mathbf{\Phi}+\frac{\lambda^2}{2\mu_0}\int dS^{(1)}\mathbf{n}^{(1)}\left(\mathbf{b}^{(1)}\times\text{curl }\mathbf{b}^{(1)}\right)\Big|_{S^{(1)}}\\&\quad+\frac{1}{2}\int dS^{(1)}\mathbf{n}^{(1)}\left(\psi^{(2)}\mathbf{B}^{(2)}\right)\Big|_{S^{(1)}}.\end{aligned}\tag{B5}$$

Taking into account that the second term of Eq. (B5) contains only tangential components of $\mathbf{b}^{(1)}$, we transform this integral in the following way:

$$\begin{aligned}&-\int dS^{(1)}\mathbf{n}^{(1)}\left(\boldsymbol{\nabla}\psi^{(2)}\times\text{curl }\mathbf{b}^{(1)}\right)\Big|_{S^{(1)}}\\&=\int dS^{(1)}\mathbf{n}^{(1)}\left[\psi^{(2)}\times\text{curl curl }\mathbf{b}^{(1)}-\text{curl }\left(\psi^{(2)}\text{curl }\mathbf{b}^{(1)}\right)\right]\Big|_{S^{(1)}}.\end{aligned}\tag{B6}$$

It is easy to show that, by applying the Stokes theorem, the last integral in Eq. (B6) vanishes (see, for example, Ref.[56]). Finally, applying the boundary condition (5a) to the normal components of the induction we obtain

$$F = \frac{1}{2\mu_0} \int dV^{(1)}\ \mathbf{b}^{(1)}\mathbf{\Phi} + \frac{1}{2} \int dS^{(1)} \left(\psi^{(2)} \mathbf{\Phi} \mathbf{n}^{(1)} \right) \Big|_{S^{(1)}} . \tag{B7}$$

Let us note that Eq. (B7) does not contain explicitly any characteristics of the magnetic media which enter only the expressions for quantities $\mathbf{b}^{(1)}$ and $\psi^{(2)}$. From Eq. (B7), the formula (19) immediately follows for the geometry considered in the paper.

The obtained general expression (B7) for the free energy is applicable for any configuration of superconductor and magnet components (for example, for multifilamentary superconductor/magnet wires or tapes) and is valid in the whole region of the vortex state in a type-II superconductor.

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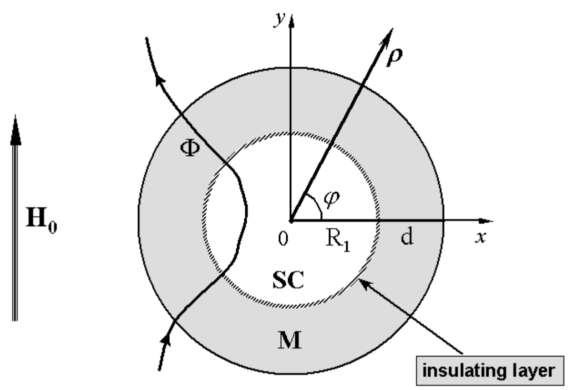

FIG. 1: Cross-sectional view of a superconductor filament covered by a coaxial cylindrical magnetic sheath and exposed to external transverse magnetic field. A plane single vortex of an arbitrary form entering a superconductor is shown.

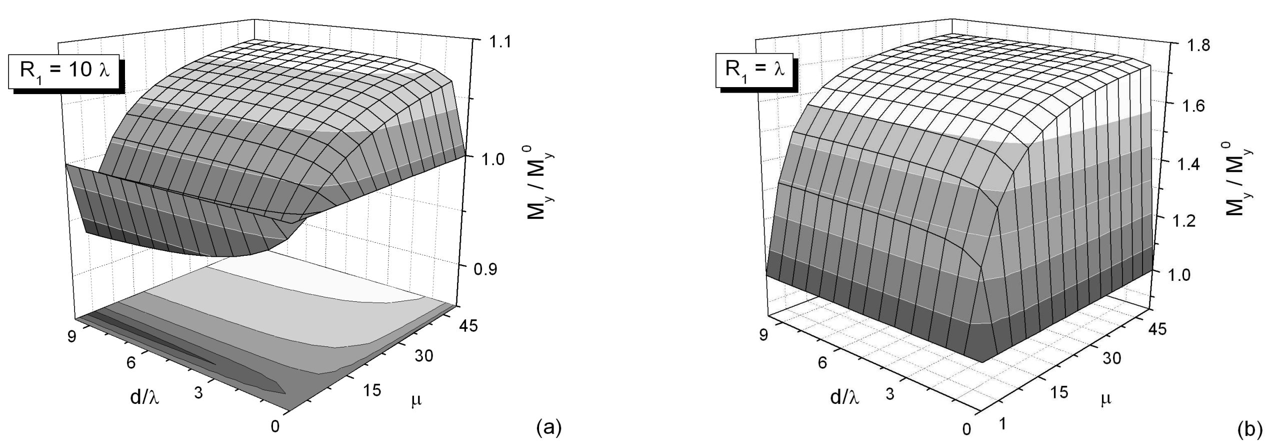

FIG. 2: The dependence of the vortex magnetic moment M_y on the relative permeability μ and on the thickness d of the magnet sheath for different values of the superconductor radius: a) $R_1 = 10\lambda$, and b) $R_1 = \lambda$.

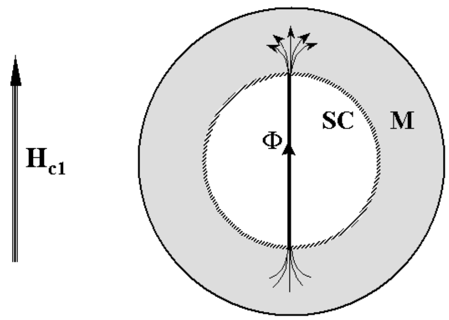

FIG. 3: Magnetic vortex lying along the diameter of a cylinder and parallel to the external transverse magnetic field.

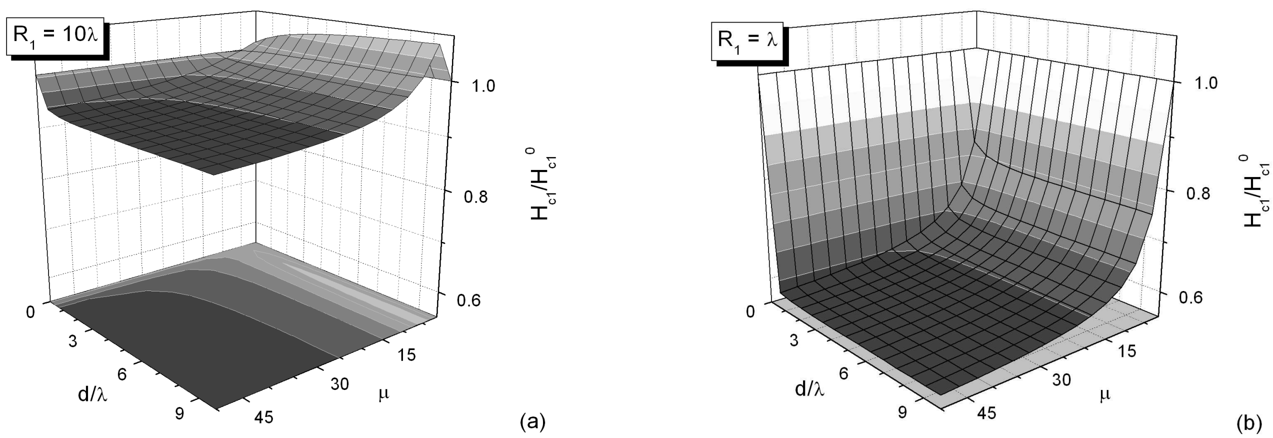

FIG. 4: The dependence of the lower critical field H_{c1} on the relative permeability μ and on the thickness d of the magnet sheath for different values of the superconductor radius: a) $R_1 = 10\lambda$, and b) $R_1 = \lambda$.

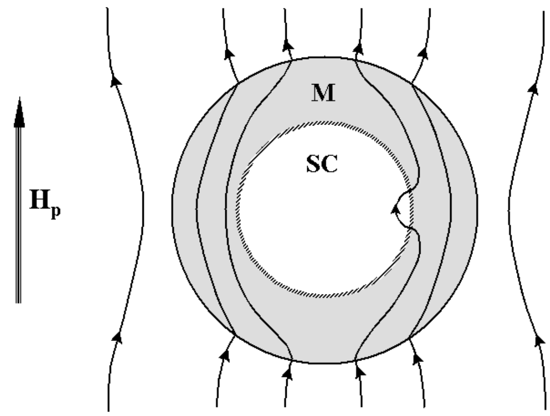

FIG. 5: Scheme of the first vortex loop nucleation at the interface between superconductor and magnet sheath in the SC/SM filament exposed to an external magnetic field.

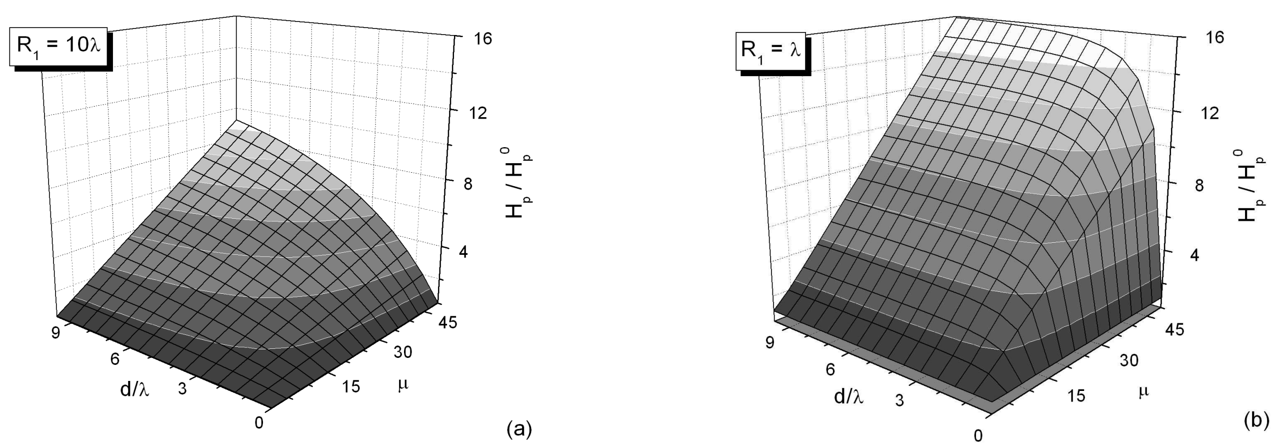

FIG. 6: The dependence of the field of the first flux penetration H_p on the relative permeability μ and on the thickness d of the magnet sheath for different values of the superconductor radius: a) $R_1 = 10\lambda$, and b) $R_1 = \lambda$.